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LETTER TO THE EDITOR

Lax representation for two-particle dynamics splitting on two tori

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Abstract. The Lax representation in terms of 2×2 matrices is constructed for separable multiply-periodic systems splitting on two tori. Hyperelliptic Kleinian functions and their reduction to elliptic functions are used.

1. Introduction

Completely integrable systems with two degrees of freedom and dynamics splitting on two tori have been largely investigated during recent years as examples of separable multiply-periodic systems. The list of such systems includes the well known integrable cases of the Hénon–Heiles system [1], several integrable cases of quartic potentials [2], the motion of a particle in the Coulomb potential and in external uniform field, the Chaplygin top [3], etc. A Lax representation for these systems can be readily constructed in terms of a direct product of Lax operators [1], one for each splitting tori, as first proposed in [4]. This approach for a system of two particles leads to 4×4 Lax representations (see e.g. [1, 2]), thus making the quantization of the above systems much more difficult to perform. In order to simplify the quantum problem it would be more convenient to use Lax representations in terms of 2×2 matrices. The problem of the existence of such representations for the above systems is still open.

The aim of the present paper is to show how to construct Lax representations in terms of 2×2 matrices for dynamics splitting on two tori. The main idea is to use a hyperelliptic curve of genus two, which is a N-sheeted cover of two given elliptic curves. Such covers are known to exist for any $N > 1$ and for arbitrary tori (see e.g. [5–7]). It is clear that, if the hyperelliptic curve is associated with a Hamiltonian system for which a 2×2 Lax representation is known, one can readily construct a similar representation for the two tori dynamics simply by using the transformation induced by the covering. To illustrate this approach we take as a working example the integrable cases of the Hénon–Heiles system [8]. The possibilities of the generalization of this approach to the system with more than two degrees of freedom is briefly discussed at the end of the paper.

2. Reduction

Consider the hyperelliptic curve $V = (y, z)$ of genus two,

$$y^2 = 4z^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \tag{1}$$

with $\lambda_i \in C$, chosen in such a way that (1) takes the form

$$w^2 = z(z-1)(z-\alpha)(z-\beta)(z-\alpha\beta). \tag{2}$$

The curve (2) gives a two-sheeted covering of two tori $\pi_\pm : V = (w, z) \to E_\pm = (\eta_\pm, \xi_\pm)$,

$$\eta_\pm^2 = \xi_\pm(1-\xi_\pm)(1-k_\pm^2\xi_\pm) \tag{3}$$

with Jacobi moduli

$$k_\pm^2 = -\frac{(\sqrt{\alpha} \mp \sqrt{\beta})^2}{(1-\alpha)(1-\beta)}. \tag{4}$$

Equation (4) can be inverted as

$$\alpha + \beta = 2\frac{k_+^2 + k_-^2}{(k'_+ - k'_+)^2} \qquad \alpha\beta = \left(\frac{k'_+ + k'_-}{k'_+ - k'_+}\right)^2 \tag{5}$$

where $k'_\pm$ are additional Jacobian moduli, $k_\pm^2 + k'^2_\pm = 1$. Explicitly, the covers $\pi_\pm$ are given by

$$\eta_\pm = -\sqrt{(1-\alpha)(1-\beta)}\frac{z \mp \sqrt{\alpha\beta}}{(z-\alpha)^2(z-\beta)^2}w \tag{6}$$

$$\xi_+ = \xi_- = \frac{(1-\alpha)(1-\beta)z}{(z-\alpha)(z-\beta)}. \tag{7}$$

Let $(y_1, x_1), (y_2, x_2)$ be arbitrary points on a symmetric degree $V \times V$. The Jacobi inversion problem is the problem of finding this point as a function $u = (u_1, u_2)$ from the equations

$$\int_{x_0}^{x_1} \frac{\mathrm{d}z}{w} + \int_{x_0}^{x_2} \frac{\mathrm{d}z}{w} = u_1 \tag{8}$$

$$\int_{x_0}^{x_1} \frac{z\,\mathrm{d}z}{w} + \int_{x_0}^{x_2} \frac{z\,\mathrm{d}z}{w} = u_2. \tag{9}$$

We write

$$\int_{x_0}^{x_1} \frac{z-\sqrt{\alpha\beta}}{w}\,\mathrm{d}z + \int_{x_0}^{x_2} \frac{z-\sqrt{\alpha\beta}}{w}\,\mathrm{d}z = u_+ \tag{10}$$

$$\int_{x_0}^{x_1} \frac{z+\sqrt{\alpha\beta}}{w}\,\mathrm{d}z + \int_{x_0}^{x_2} \frac{z+\sqrt{\alpha\beta}}{w}\,\mathrm{d}z = u_- \tag{11}$$

with

$$u_\pm = -\sqrt{(1-\alpha)(1-\beta)}(u_2 \mp \sqrt{\alpha\beta}u_1). \tag{12}$$

We can reduce the hyperelliptic integrals in (10) and (11) to elliptic ones by using the formula

$$\frac{\mathrm{d}\xi_\pm}{\eta_\pm} = -\sqrt{(1-\alpha)(1-\beta)}(z \mp \sqrt{\alpha\beta})\frac{\mathrm{d}z}{w}. \tag{13}$$

Let us introduce the coordinates (see [9])

$$\begin{aligned} X_1 &= \mathrm{sn}(u_+, k_+)\mathrm{sn}(u_-, k_-) \\ X_2 &= \mathrm{cn}(u_+, k_+)\mathrm{cn}(u_-, k_-) \\ X_3 &= \mathrm{dn}(u_+, k_+)\mathrm{dn}(u_-, k_-) \end{aligned} \tag{14}$$

where $\mathrm{sn}(u_\pm, k_\pm)$, $\mathrm{cn}(u_\pm, k_\pm)$, and $\mathrm{dn}(u_\pm, k_\pm)$ denote usual Jacobi elliptic functions [10]. Applying the addition theorem for Jacobi elliptic functions,

$$\mathrm{sn}(u_1 + u_2, k) = \frac{s_1^2 - s_2^2}{s_1 c_2 d_2 - s_2 c_1 d_1}$$
$$\mathrm{cn}(u_1 + u_2, k) = \frac{s_1 c_1 d_2 - s_2 c_2 d_1}{s_1 c_2 d_2 - s_2 c_1 d_1}$$
$$\mathrm{dn}(u_1 + u_2, k) = \frac{s_1 d_1 c_2 - s_2 d_2 s_1}{s_1 c_2 d_2 - s_2 c_1 d_1}$$

where $s_i = \mathrm{sn}(u_i, k)$, $c_i = \mathrm{cn}(u_i, k)$, $d_i = \mathrm{dn}(u_i, k)$, $i = 1, 2$, we can write equations (14) in the form

$$X_1 = -\frac{(1-\alpha)(1-\beta)(\alpha\beta + \wp_{12})}{(\alpha+\beta)(\wp_{12} - \alpha\beta) + \alpha\beta\wp_{22} + \wp_{11}} \qquad (15)$$
$$X_2 = -\frac{(1+\alpha\beta)(\alpha\beta - \wp_{12}) - \alpha\beta\wp_{22} - \wp_{11}}{(\alpha+\beta)(\wp_{12} - \alpha\beta) + \alpha\beta\wp_{22} + \wp_{11}} \qquad (16)$$
$$X_3 = \frac{\alpha\beta\wp_{22} - \wp_{11}}{(\alpha+\beta)(\wp_{12} - \alpha\beta) + \alpha\beta\wp_{22} + \wp_{11}}. \qquad (17)$$

Here $\wp_{ij}$ are Kleinian $\wp$-functions which solve the Jacobi inversion problem and are expressed in terms of (y_1, x_1), (y_2, x_2) as follows,

$$\wp_{22} = x_1 + x_2 \qquad \wp_{12} = -x_1 x_2 \qquad \wp_{11} = \frac{F(x_1, x_2) - 2y_1 y_2}{4(x_1 - x_2)^2}$$

and

$$F(x_1, x_2) = \sum_{k=0}^{k=2} x_1^k x_2^k (2\lambda_{2k} + \lambda_{2k+1}(x_1 + x_2)) \qquad (18)$$

with λ's calculated from (2). The Kleinian $\wp$-functions are known to be a natural generalization of the Weierstrass elliptic functions and can then be expressed through the second logarithmic derivative of the Kleinian σ-function,

$$\wp_{ij}(u) = -\frac{\partial^2 \ln \sigma(u)}{\partial u_i \partial u_j} \qquad i, j = 1, 2$$

(for details see [5, 11]). The three functions $\wp_{22}$, $\wp_{12}$, $\wp_{11}$ are algebraically dependent and are coordinates for the so-called Kummer surface which is a quartic surface in C^3. For later convenience we remark that the formulae (15)–(17) can be inverted as

$$\wp_{11} = (B-1)\frac{A(X_2 + X_3) - B(X_3 + 1)}{X_1 + X_2 - 1} \qquad (19)$$
$$\wp_{12} = (B-1)\frac{1 + X_1 - X_2}{X_1 + X_2 - 1} \qquad (20)$$
$$\wp_{22} = \frac{A(X_2 - X_3) + B(X_3 - 1)}{X_1 + X_2 - 1} \qquad (21)$$

where $A = \alpha + \beta$, $B = 1 + \alpha\beta$.

3. Lax representation

Let us consider the following equations for the four-indexed functions $\wp$:

$$\wp_{2222} = 6\wp_{22}^2 + 4\wp_{12} + \lambda_4\wp_{22} + \tfrac{1}{2}\lambda_3 \tag{22}$$
$$\wp_{1222} = 6\wp_{22}\wp_{12} - 2\wp_{11} + \lambda_4\wp_{12} \tag{23}$$

with λ_3 and λ_4 arbitrary. The first equation, after u_2 differentiation, is the standard KdV equation written with respect to the function $\wp_{22}$ while the second equation represents the stationary flow for the two gap KdV solution ($\wp_{22}$) of the third vector field of the KdV hierarchy. As is well known, equations (22) and (23) can be written in the Lax form [12],

$$\frac{\partial \mathbf{L}}{\partial t} = [\mathbf{M}, \mathbf{L}] \qquad \mathbf{L} = \begin{pmatrix} V & U \\ W & -V \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix}. \tag{24}$$

Here we take the elements of the matrices $\mathbf{L}$ and $\mathbf{M}$ to be polynomials in x of the form

$$U = x^2 - \wp_{22}x - \wp_{12} \tag{25}$$
$$V = -\frac{1}{2}\frac{\partial U}{\partial u_2} \tag{26}$$
$$W - -\frac{1}{2}\frac{\partial^2 U}{\partial u_2^2} + UQ \tag{27}$$
$$Q = x + 2\wp_{22} + \tfrac{1}{4}\lambda_4. \tag{28}$$

The discriminant curve $\det(\mathbf{L} - y\mathbf{E}) = 0$ ($\mathbf{E}$ is the 2×2 unit matrix) then has the form of equation (1) with λ_4, λ_3, λ_0 arbitrary and λ_2, λ_1 chosen as the level set of the integrals of motion:

$$-\lambda_2 = -\wp_{222}^2 + 4\wp_{11} + \lambda_3\wp_{22} + 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + \lambda_4\wp_{22}^2 \tag{29}$$
$$-\tfrac{1}{2}\lambda_1 = -\wp_{222}\wp_{221} + 2\wp_{12}^2 - 2\wp_{11}\wp_{22} + \tfrac{1}{2}\lambda_3\wp_{12} + 4\wp_{12}\wp_{22}^2 + \lambda_4\wp_{12}\wp_{22}. \tag{30}$$

The following proposition represents the main result of the paper.

Proposition. Let

$$U = x^2 - \frac{A(X_2 - X_3) + B(X_3 - 1)}{X_1 + X_2 - 1}x + (B - 1)\frac{X_1 - X_2 + 1}{X_1 + X_2 - 1} \tag{31}$$
$$Q = x + 2\frac{A(X_2 - X_3) + B(X_3 - 1)}{X_1 + X_2 - 1} + A + B \tag{32}$$

where X_i are the coordinates given in (14) and $A = \alpha + \beta$, $B = \alpha\beta + 1$ are expressed in terms of Jacobian moduli $k_\pm$ according to (5). Then the Lax equation (24) is equivalent to the equations for Jacobi elliptic functions,

$$\frac{\mathrm{d}}{\mathrm{d}u_\pm}\mathrm{sn}(u_\pm; k_\pm) = \sqrt{(1 - \mathrm{sn}^2(u_\pm; k_\pm)(1 - k_\pm^2\mathrm{sn}^2(u_\pm; k_\pm))}. \tag{33}$$

To prove this statement one can expand $\mathrm{sn}(u_\pm; k_\pm)$ around $u_\pm = 0$ to obtain equation (33) from (24) with superscripts '$\pm$'. We remark that a direct substitution of (19)–(21) into the equations of motion (22) and (23) would be quite involved even for symbolic calculations on a computer.

4. An example: Lax representation for the integrable cases of the Hénon–Heiles system

Let us apply the above result to the integrable cases of the Hénon–Heiles system (see e.g. [8]). One of them (case (ii) in the terminology of [8]) is isomorphic to the fifth-order stationary KdV flow, thus giving a Lax representation in terms of 2×2 matrices. The other two cases—cases (i) and (iii)—are isomorphic to the fifth stationary flows of the Sawada–Kotera and Kaup–Kuperschmidt equations, respectively. They both lead to Lax representations in terms of 3×3 matrices [8]. The 4×4 Lax representation is derived in [1]. Let us show how to construct the 2×2 Lax representation for cases (i) and (iii).

Consider first the integrable case (i). The Hamiltonian H and second integral motion K have the form

$$H = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + q_1q_2^2 + \tfrac{1}{3}q_1^3 + a(q_1^2 + q_2^2) \tag{34}$$

$$K = p_1p_2 + \tfrac{1}{3}q_2^3 + q_2q_1^2 + 2aq_1q_2. \tag{35}$$

The Hamiltonian system is separated in Cartesian coordinates, $q_{1,2} = \tilde{Q}_1 \pm \tilde{Q}_2$, $p_{1,2} = \tilde{P}_1 \pm \tilde{P}_2$ and the dynamics is splitting to two tori

$$\begin{aligned} \tilde{P}_1^2 &= -\tfrac{4}{3}\tilde{Q}_1^3 - 2a\tilde{Q}_1^2 + \tfrac{1}{2}(\tilde{H} + \tilde{K}) \\ \tilde{P}_2^2 &= -\tfrac{4}{3}\tilde{Q}_2^3 - 2a\tilde{Q}_2^2 + \tfrac{1}{2}(\tilde{H} - \tilde{K}) \end{aligned} \tag{36}$$

where $\tilde{H} = \tilde{P}_1^2 + \frac{4}{3}\tilde{Q}_1^3 + \frac{4}{3}\tilde{Q}_3^3 + 2a(\tilde{Q}_1^2 + \tilde{Q}_2^2)$, $\tilde{K} = \tilde{P}_1^2 - \tilde{P}_2^2 + \frac{4}{3}\tilde{Q}_1^3 - \frac{4}{3}\tilde{Q}_3^3 + 2a(\tilde{Q}_1^2 - \tilde{Q}_2^2)$. By passing from (36) to the standard form of the elliptic curve (33) we find

$$\wp^{\pm}\left(\frac{\mathrm{i}t}{\sqrt{3}}\right) = \tfrac{1}{2}(q_1(t) \pm q_2(t) + a) \tag{37}$$

with $\wp^{\pm}$ standard Weierstrass elliptic functions with moduli $e_i^{\pm}$, $i = 1, 2, 3$ satisfying the equations

$$4e_1^{\pm}e_2^{\pm}e_3^{\pm} = a^3 - \tfrac{3}{2}(\tilde{H} \pm \tilde{K}) \qquad 8(e_1^{\pm}e_2^{\pm} + e_1^{\pm}e_3^{\pm} + e_2^{\pm}e_3^{\pm}) + \tfrac{3}{2}a^2 = 0. \tag{38}$$

The Lax representation (24) is then valid for the system with

$$\begin{aligned} X_1 &= \sqrt{\frac{2e_1^+ - 2e_3^+}{q_1 + q_2 + a - 2e_3^+}}\sqrt{\frac{2e_1^- - 2e_3^-}{q_1 - q_2 + a - 2e_3^-}} \\ X_2 &= \sqrt{\frac{q_1 + q_2 + a - 2e_1^+}{q_1 + q_2 + a - 2e_3^+}}\sqrt{\frac{q_1 - q_2 + a - 2e_1^-}{q_1 - q_2 - 2e_3^-}} \\ X_3 &= \sqrt{\frac{q_1 + q_2 + a - 2e_2^+}{q_1 + q_2 + a - 2e_3^+}}\sqrt{\frac{q_1 - q_2 + a - 2e_2^-}{q_1 - q_2 + a - 2e_3^-}}. \end{aligned}$$

As shown in [13], the integrable case (iii) is linked to case (i) by means of a canonical transformation. The corresponding 2×2 Lax representation can then be derived from the representation of case (i) by means of this transformation.

5. Concluding remarks

In closing this paper we make the following remark. Equip the curve by the canonical basis of cycles A_1, A_2, B_1, B_2 and normalize the holomorphic differentials $\mathrm{d}v_i =$

$(c_{i1} + zc_{i2})\,\mathrm{d}z/w(z)$, $i = 1, 2$, in such a way that the Riemann matrix $\mathbf{\Omega}$ has the following form

$$\mathbf{\Omega} = \begin{pmatrix} \oint_{A_1} \mathrm{d}v_1 & \oint_{A_2} \mathrm{d}v_1 & \oint_{B_1} \mathrm{d}v_1 & \oint_{B_2} \mathrm{d}v_1 \\ \oint_{A_1} \mathrm{d}v_2 & \oint_{A_2} \mathrm{d}v_2 & \oint_{B_1} \mathrm{d}v_2 & \oint_{B_2} \mathrm{d}v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau_{11} & \tau_{12} \\ 0 & 1 & \tau_{12} & \tau_{22} \end{pmatrix}. \quad (39)$$

It is known (see e.g. [6, 7, 9]) that the curve (1) covers N-sheetedly two tori if and only if the Riemann matrix $\mathbf{\Omega}$ can be transformed by some linear transformation of the basis cycles to the form

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \frac{1}{N} \\ \frac{1}{N} & \tau_{22} \end{pmatrix}$$

where the positive integer N is also called the Picard number. The condition for the matrix $\boldsymbol{\tau}$ to be transformed to the form given above is that $\boldsymbol{\tau}$ belongs to the Humbert surface H_N

$$\begin{aligned} H_N = \{&\alpha\tau_{11} + \beta\tau_{12} + \gamma\tau_{22} + \delta(\tau_{12}^2 - \tau_{11}\tau_{22}) + \varepsilon = 0, \\ &\alpha, \beta, \gamma, \delta, \varepsilon \in Z, \beta^2 - 4(\alpha\gamma + \varepsilon\delta) = N^2\}. \end{aligned}$$

The case considered in this paper corresponds, among the infinite transformations of Nth order which permit one to reduce the dynamics of a two-particle system associated with the N-sheeted covering of tori, just to the case $N = 2$. It is clear, however, that the above analysis can be extended to curves of high genus.

These arguments were used in [7] to describe elliptic potentials of the Schrödinger equation, which were also studied in the framework of spectral theory [14–16].

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